FRIEDRICH-SCHILLER-UNIVERSITÄT JENA

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Stochastic Analysis (SS 2019)

Exercise sheet 5

Submission: 14.05.2019

Problem 1 (4 Points)

Let $(B_t)_{t\geq 0}$ be a Brownian motion. Let $a > 0 > b$ and define

$$\tau_a := \inf\big\{t \geq 0 \,|\, B_t = a\big\}, \quad \tau_b := \inf\big\{t \geq 0 \,|\, B_t = b\big\},$$
$$\text{and } \tau_{a,b} := \inf\big\{t \geq 0 \,|\, B_t \notin (b, a)\big\} = \tau_a \wedge \tau_b.$$

(a) Show that $\mathbb{P}[\tau_{a,b} < \infty] = 1$, $\mathbb{P}[B_{\tau_{a,b}} = a] = \mathbb{P}[\tau_a < \tau_b] = \frac{b}{b-a}$ and that $\mathbb{E}[\tau_{a,b}] = -ab$.
Hint: Use that $t \mapsto B_t$ and $t \mapsto B_t^2 - t$ are martingales.

(b) Deduce that $\mathbb{P}[\tau_a < \infty] = 1$, but $\mathbb{E}[\tau_a] = \infty$.

Problem 2 (3 Points)

Let $(B_t)_{t\geq 0}$ be a Brownian motion. For $a > 0$ let $\tau_a = \inf\{t \geq 0 \,|\, B_t = a\}$. For $\lambda > 0$ choose a suitable martingale to show that

$$\mathbb{E}\left[\exp(-\lambda\tau_a)\right] = \exp\big(-a\sqrt{2\lambda}\big).$$

Problem 3 (4 Points)

Let $(B_t)_{t\geq 0}$ be a Brownian motion and let $\mu > 0$.

(a) Let $S_t = \exp\big(2\mu(B_t - \mu t)\big)$. Conclude that (S_t) is a martingale and that $\lim_{t\to\infty} S_t = 0$, $\mathbb{P}$-a.s.

(b) For $a > 0$ consider

$$\sigma_a := \inf\{t \geq 0 \,|\, B_t - \mu\, t \geq a\}.$$

Show that $\mathbb{P}[\sigma_a < \infty] = e^{-2\mu a}$.

Problem 4 - Brownian Bridge **(4 Points)**

Let $(B_t)_{t\geq 0}$ be a Brownian motion and define

$$Y_t = B_t - tB_1, \quad t \in [0,1].$$

$(Y_t)_{t\geq 0}$ is a so-called Brownian bridge from zero to zero.

(a) Compute $\mathbb{E}[Y_t]$ and $\mathrm{Cov}(Y_s, Y_t)$ for $s, t \in [0,1]$.

(b) Show that Y_t and B_1 are independent for all $t \in [0,1]$.

Total: 15 Points

Terms of submission:

- Solutions can be submitted in groups of at most 2 students.
- Please submit at the beginning of the lecture or until 9:50 a.m. in room 3523, Ernst-Abbe-Platz 2.